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Subgroup Graphs of Finite Groups

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Abstract

Let G be a finite group with the set of subgroups of G denoted by $S(G)$, then the subgroup graphs of G denoted by $\Gamma(G)$ is a graph which set of vertices is $S(G)$ such that two vertices $H, K \in S(G)$ $(H /= K)$ are adjacent if either H is a subgroup of K or K is a subgroup of H. In this paper, we introduce the Subgroup graphs Γ associated with G. We investigate some algebraic properties and combinatorial structures of Subgroup graph $\Gamma(G)$ and obtain that the subgroup graph $\Gamma(G)$ of G is never bipartite. Further, we show isomorphism and homomorphism of the Subgroup graphs of finite groups.

Keywords: subgroup, graph, finite group

1 Introduction

One of the mathematical tools for studying symmetries of object is group theory, hence, several structures in the field of algebra are depicted through groups. This mathematical concept has evolved rapidly since it discovery in the sixteenth century. According to [1], the rebirth of the axiomatic method and the view of mathematics as a human activity in the nineteenth century forms the major development that change the bearing on the evolution of group theory as a mathematical concept. [1], further noted that the evolution had caused the previous classical algebra polynomial equations transited to the modern algebra of axiomatic systems of the nineteenth century. Meanwhile, this concept has been applied in the field of physics, chemistry and biology, (see [2], [3], [4], [5]) for details.

In the same vein, in the last two decades, many studies have related graphs to group theory, providing a more easier way to visualize the concept of group; this relation brings together two important branches of mathematics, and has opened up a new wave of research with a better understanding of the fields. Many years after Euler's research work on the bridges of Konigsberg city, Cayley [6] used the generators of a finite group G to define a graphical structure called the Cayley graph of finite group G, he further showed that every group of order n can be represented by a strongly connected diagraph of n vertices [7]. Afterwards, in the last few decades, his view of diagraph has since been extended to different and modified graph of algebraic structures. Hence, more algebraic studies through the properties of these modified graphs have become topics of interest to many around the globe (See [8], [9], [10], [11], [12], [13], [14], [15]).

This study, the subgroup graph of finite groups $\Gamma(G)$ like [8], [9], [10], [11], [12], [13], [14], [15], will focus on finite groups G, however, the choice of it vertex set $V(\Gamma(G))$, is the subgroups $S(G)$ of G. In the literature, vertex set of graphs of finite groups are always the elements n G, a deviation from this norm is the motivation for this study.

1.1. **Preliminaries**. We state some known and useful results which will be needed in the proof of our main results and understanding of this paper. For the definitions of the basic terms and results given in this section ([16], [17], [18], [19], [20], [21], [22], [23]). A *graph* Γ is a combinatorial structure formed by finite non-empty set (V, E), where V is the set of vertices viewed as points and E is the set of edges viewed as line joining the points. The cardinality of $V(\Gamma)$ is called the *order* of Γ while the cardinality of $E(\Gamma)$ is called the *size* of Γ. The degree of a vertex x in a graph Γ denoted by $\delta(x)$ is the number of edges incident to it, that is the number of edges connecting x. A graph is said to have *parallel edges* if there are more than one edges which join the same pair of

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distinct vertices. A *loop* on the other hand is an edge that joins a vertex to itself while a *walk* of length $k \leq n$ in a graph Γ with vertex set $V\,(\Gamma)$ consist of an alternating sequence of vertices and edges consecutive elements of which are incident, that begins and ends with a vertex.

Definition 1.1. [20] A *complete graph* is a simple undirected graph in which has at least one vertex and every arbitrary pair of distinct vertices is joint by a unique edge. while a connected graph on the other hand is a graph where there is an edge between every pair of vertices.

Remark 1.2. Note that every complete graph is necessary connected but connected graphs are not necessary complete.

Definition 1.3. [20] A walk in a connected graph that visits every vertex of the graph exactly once without repeating the edges is called *Hamiltonian path*. If this walk starts and ends at the same vertex, the walk is called a Hamiltonian circuit or cycle.

Remark 1.4. A graph that contains a *Hamiltonian* cycle is said to be *Hamiltonian*.

Theorem 1.5. *(Sylow's First Theorem)* [21] *Let G be a finite group of order $p^r q$, where p is a prime, r and q are positive integers and $gcd(p,q) = 1$. Then G has a subgroup of order p^k for all k satisfying* $0 \leq k \leq r$.

Definition 1.6. [20] In graph theory, a *regular graph* is a graph where each vertex has the same number of neighbors, i.e. every vertex has the same degree or valency; a regular graph can be an x-regular graph where every vertex of the graph have the same degree x.

Definition 1.7. [20] The distance between two vertices $x, y\, V\,(\Gamma(G))$ is the *length* of the shortest path between x and y and it's denoted by $\delta(x,y)$. *Eccentricity* $\delta(x)$ of a vertex x in a graph is define as $\delta(x) = max\,\{\delta(x,y) : x, y\, V\,(G)\,\}$. The minimum and maximum eccentricity in a graph are called *radius*, rad(G) and *diameter*, diam(G) of the graph respectively.

Lemma 1.8. [20] *Let G and G' be any two finite groups. If $G \cong G'$, then $S'(G) \cong S'(G')$ and $S(G) \cong S(G')$. But the converse is not true.*

Definition 1.9. [21] A *group consists* of a set G with a binary operation $*$ on G satisfying the following four conditions:

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1. Closure: $\forall a, b \in G$, we have $a * b \in G$.
2. Associativity: $\forall a, b, c \in G$, we have $a(b * c) = (a * b)c$.
3. Identity: There is an element $e \in G$ satisfying $e * a = a * e = a$ for all $a\ G$.
4. Inverse: For all $a \in G$, there is an element $a^{-1} \in G$ satisfying $a * a^{-1}\ a^{-1} * a = e$. (where e is as in the Identity Law)

Definition 1.10. [20] A finite group is a group containing finite number of elements. The order of a finite group G is the number of elements in G.

Definition 1.11. [21] A subset H of a group G is called a subgroup if it forms a group in its own right with respect to the same operation on G.

Definition 1.12. [21] Let G_1 and G_2 be groups. A homomorphism from G_1 to G_2 is a map θ which preserves the group operation.

Definition 1.13. [21] A subgroup H of G is said to be a normal subgroup if it is the kernel of a homomorphism. Equivalently, H is a normal subgroup if its left and right cosets coincide: $aH = Ha$ for all $a \in G$. We write "H is a normal subgroup of G" as $H \trianglelefteq G;\ if\ H \neq G$, we write $H \triangleleft G$. If H is a normal subgroup of G, we denote the set of (left or right) cosets by G/H. We define an operation on G/H by the rule $(Ha)(Hb) = Hab$ for all $a, b \in G$.

Definition 1.14. [24] Let a and b be elements of a group G such that $[a, b]$ yields an element of G and is defined by $[a, b] = a^{-1}.b^{-1}.a.b$, the collections of arbitrary $[a, b]$ in G forms the commutator subgroup of G.

Lemma 1.15. [16] Suppose $a, b, c \in G$ and e is a positive integer. Then

1. $[a, b] = [b, a]^{-1}$.
2. $[a, bc] = [a, c][a, b]^c$.
3. $[ab, c] = [a, c]^b[b, c]$.
4. $[a, b^e] = \prod_{i=0}^{e-1} [a, b]^{b^i}$.

Lemma 1.16. [25] Let G be a group with $P\ a\ p -$ subgroup of G and $T\ a\ p' -$ subgroup of $N_G(P\ )$. Set $R = [T, P\ ]$, and the following holds

1. $[T, R] = R$, *and if* P *is abelian, then* $P = R * C_P\ (T\ )$.
2. *If* R *is abelian, then the minimal number of elements needed to generate* $R;\ d(R) \leq 2$ *and* $t \in T\ ' - C_\tau\ (P\ )$, *then* $[t, R] = R$..

3. *If R is abelian and the minimal number of elements needed to generate R; $d(R) \leq 2$, there exists $t \in T$ such that $[t, R] = R$.*
4. *If R is cyclic, then $T/C_\tau(P)$ is cyclic*

Lemma 1.17. [25] *Suppose $G = \langle x, y, A \rangle$ is a finite group with $A \geq G'$ abelian and $a = [x, y]$ of order n. If C is the commutator subgroup of G, then $\{a^e b \mid b \in [G, A], (e, n) = 1\} \subseteq C$.*

Theorem 1.18. [*25] Let G be a $p-$group with G' abelian and $d(G') \leq 2. Set\ C = CG(G'/\Phi(G'))$.*

1. *There exists $y \in G$ with $G' = [y, C]$. In particular, if $C < G$, then y can be taken to be any element of $G - C$.*
2. *Let $[y, C] = G'$. If either $p \neq 2$ or $[G', C] \leq \Omega^2 G'[y, G']$, then $G' = \Gamma_y(C)$.*
3. *G' is equal to the commutator subgroup of G.*

Lemma 1.19. [25] *Let P be a $p-$subgroup of G, t a $p'-$element of $N_G(P)$, and $s \in P$. If $[t, P]$ is abelian, then $\Gamma_{ts}(P) = [t, P]\Gamma_s(P)$.*

Theorem 1.20. [26] *Let G be a group and let H, K be two subgroups of G and define $= hk : h \in H, k \in K$, then*

1. *If both H and K are normal in G, then $H \cap K$ is also a normal subgroup of G.*
2. *If H alone is normal in G, then $H \cap K$ is a normal subgroup of K.*
3. *If H is normal in G, then $HK = KH$ and HK is a normal subgroup of G.*
4. *If both H and K are normal in G, then HK is a normal subgroup of G.*

Theorem 1.21. *(Sylow's theorems) Let G be a group of order $p^\alpha m$, where p is a prime, $m \geq 1$, and p does not divide m. Then:*

1. *$Sylp \neq \emptyset$, that is $Sylow$ $p-$subgroups exist.*
2. *All $Sylow$ $p-$subgroups are conjugate in G.*
3. *Any $p-$subgroup of G is contained in a $Sylow$ $p-$subgroup.*
4. *$n_p(G) \equiv 1 \bmod p$.*

Lemma 1.22. *(Lagrange's theorem) [22] Let H be a subgroup of a finite group G. Then the order of H divides the order of G.*

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2 Research Methodology

This article is not a variable base research, however, well known algebraic definitions and results were used to investigate the algebraic and combinatorial properties of the subgroup graph of finite groups.

3 Results and Discussion

The Subgroup graphs of finite groups is introduce in this section. We begin with the definition and notion of the Subgroups graph of a finite group.

Definition 3.1. Let G be a finite group and $S(G)$ be the set of subgroups of G. Then the Subgroup graph of G is the graph $\Gamma(G)$ with vertex set $V(\Gamma(G) = S(G)$ and the edge set

$$E(\Gamma(G)) = \{\{H, K\} : H \neq K, H \leq K \text{ or } K \leq G\}.$$

Remark 3.2. Let G be a finite group of order n, some clear consequences of the definition of subgroup graphs of finite groups $\Gamma(G)$ are

1. The subgroup graph $\Gamma(G)$ is a simple graphs, thus, there are no loops nor multiple edges.
2. The trivial subgroups of G are adjacent to every other vertices on $\Gamma(G)$.
3. Since the trivial subgroups of G are adjacent to every other vertices on $\Gamma(G)$, then the graph is connected.
4. The $\Gamma(G)$ has a diameter and radius of 1 if $|G| > 2$.

Below, we give an example of Subgroups graph.

Example 3.3. Let G = the group of integer modulo 6 under addition $(\mathbb{Z}_6, +)$, then the following is the undirected Subgroups graph of $(\mathbb{Z}_6)$. See Figure 1.

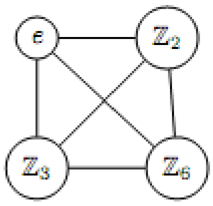

Figure 1. Subgroups graph of Z_6.

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Theorem 3.4. [20] *A simple graph is bipartite if and only if it does not have any odd cycle.*

Remark 3.5. Let G be a finite group, the subgroup graph $\Gamma(G)$ of G is never bipartite.

Theorem 3.6. Let G be a finite group of prime order and let $S(G)$ be the set of subgroups of G, then the subgroups graph of G is a straight line with only two vertices.

Proof: Let G be a finite group of prime order and let $S(G)$ be the set of subgroups of G, then by Lemma 1.22, the order of every $H \in S(G)$ divides the order of G but the order of G is a prime which can only be divided by itself of 1. Thus the only subgroups of this group G are $H = \{\{e\}\ and\ G\ itself\}$; and they are adjacent.

Remark 3.7. Let G be a finite group and let $\Gamma(G)$ be the subgroups graph of G. Then the vertex set $V\ (\Gamma(G)) \geq 2$ and edge set $E(\Gamma(G)) \neq \emptyset$; therefore, the subgroups graph of a finite group G can never be empty.

Theorem 3.8. Let G be a group and let H, K be two subgroups of G and let $\Gamma(G)$ be the subgroup graph of G. Define $HK = \{hk: h \in H, k \in K\}$, then

1. If both H and K are normal in G, then $H \cap K$ is also a vertex on $\Gamma(G)$.
2. If H alone is normal in G, then $H \cap K$ is a adjacent to vertex K on $\Gamma(G)$.
3. If H is normal in G, then $HK = KH$ and HK is also a vertex on $\Gamma(G)$.
4. If both H and K are normal in G, then HK is also a vetex on $\Gamma(G)$.

Proof: From Theorem 1.20, the results follows.

Theorem 3.9. Let G be a finite group of non prime order $n > 2$, then the subgroup graph of G ($\Gamma(G)$) is never a star graph.

Proof: Suppose on the contrary, let the subgroup graph of a finite group of non prime order $n > 2$ be a star graph; then it implies that all other $H \in S(G)$ are subgroup to only an arbitrary subgroup $K \subseteq G$, but by Remark 3.2(2), every group has two trivial subgroups which are adjacent to all other $H \in S(G)$. So, the graph can not be a star graph, since, there is more than one vertex that is adjacent to all the vertices of $\Gamma(G)$.

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Theorem 3.10. Let G and G' be two isomorphic finite groups. Then the subgroup graph of G is isomorphic to subgroup graph of $G'(\Gamma(G) \cong \Gamma(G'))$.

Proof: Suppose G and G' are two isomorphic finite groups, then from Lemma 1.8, $S(G) \cong S(G')$. Thus, $(\Gamma(G) \cong \Gamma(G'))$.

Theorem 3.11. Let G be a group of order $p^{\alpha}\, m$, where p is a prime, $m \geq 1$ and p does not divide m. Suppose $\Gamma(G)$ is the subgroup graph of G, $V\,(\Gamma(G))$ and $E(\Gamma(G))$ are the vertex and edge sets of $\Gamma(G)$ respectively. Then:

1. There must exist Sylow p-subgroups of $G; H, K \in V(\Gamma(G))$ which are conjugate in G and are adjacent to each other on the subgroup graph of G.
2. There is a vertex H on the subgroup graph of G that is adjacent to a vertex K that is a Sylow p-subgroup of G.

Proof:

1. To show that there must exist Sylow p-subgroups of $G; H, K \in V(\Gamma(G))$ which are conjugate in G and are adjacent to each other on the subgroup graph of G, then it suffices if we can show $H, K \in V(\Gamma(G))$ to be conjugate and subsequently establish an edge between H and K. From Theorem 1.21, it is established that the group G has some Sylow p-subgroups. So, let H be a Sylow p-subgroup of G, and let S be the set of all distinct conjugates of H. Suppose k is the order S, we need to establish that p cannot divide $k = |S|$. Since each of the $K \in S$ is a conjugate to H, it implies every element of S is in the orbit of H. So using the formula for orbit size $|S| = |orbit\ of\ H| = (G{:}\, N_G\,(H))$ (where $N_G\,(H))$ is the normalizer of H). However, Lagrange's theorem established that $|G| = (G{:}\, N\_G\,(H))$ and clearly, H is a subgroup of $N_G(H)$ and it contains p^{α} as a factor and a maximum power p can assume. Thus, $(|G|\ divides\ |N_G\,(H)|$ and $|S|$ contains no factor of p and so p does not divide $|S|$.
2. Suppose H is any p-subgroup of G, it will suffice if we can show $H \leq K\_i, i = 1, \dots, p$. Let H act on $K = \{K_i \dots K_p\}$, by conjugation. Clearly, the the orbits of this action will partition K. Suppose the distinct orbits are the orbits of $\{K_{1i} \dots K_{1p}\}$ then the orbit of $|K| = |orbit\ K_{1i}\,| + \cdots + |orbit\ K_{1p}\,|$. To compute the orbit for

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any $K_{1i} \in K$, $|orbit\ of\ K_{1i}| = (H{:}\,H \cap N_G(K_i)$, since $H \cap N_G(K_i)$ is the stabilizer of K_i under the action of H. Then the size of each orbit has to divide $|H|$, which is a power of p. Though p doesn't divide $|H|$ so there is no p dividing all the terms $|orbit\ of\ P_{1i}\ |, \dots, |\ orbit\ of\ P_{1i}\ |$ and else p would divide their sum and also K. Assume $1 = |orbit\ of\ K_{1i}\ | = (H{:}\,H \cap\ N_G(K_i)$ which means $(H = H \cap N_G(K_i)$ and since H is a $p-$subgroup $H \cap N_G(K_i) =\ H\ \cap K_i$. This implies $H = H\ \cap K_i$ so every element of H is also in K_i then $H \leq K$, therefore, the $p-$subgroup H is a subgroup of the arbitrary Sylow $p-$subgroup of G. So, H and K are adjacent of the subgroup graph $\Gamma(G)$ of G.

Theorem 3.12. Let G and G' be two finite groups and $\phi : G \to G'$ be a group homomorphism. Suppose there is an N, a normal subgroup of G and an N', a normal subgroup of G' such that N is adjacent to G on $\Gamma(G)$ the subgroup graph of G and N' is adjacent to G' on $\Gamma(G')$ the subgroup graph of G'. Then

1. $\phi(N)$ is adjacent to $\phi(G)$ on $\Gamma(\phi(G))$ the subgroup graph of $\phi(G)$.
2. $\phi^{-1}(N)$ is adjacent to G on $\Gamma(G)$ the subgroup graph of G.

Proof: Suppose G and G' are two finite groups and $\phi : G \to G'$ is a group homomorphism, if there is an N , a normal subgroup of G and an N', a normal subgroup of G' such that N is adjacent to G on $\Gamma(G)$ and N' is adjacent to G' on $\Gamma(G')$ then to show that $\phi(N)$ is also adjacent to $\phi(G)$ on $\Gamma(\phi(G))$ and $\phi^{-1}(N)$ is adjacent to G on $\Gamma(G)$; it will suffice if we can show $\phi(N)$ to be a normal subgroup of $\phi(G)$ and $\phi^{-1}(N)$ to be a normal subgroup of G respectively.

1. Let $\phi(g) \in \phi(G)$, since ϕ is a group homomorphism and N is normal in G. $\phi(g)\phi(N\ )\phi(g)^{-1}\ =\ \phi(gNg^{-1}) = \phi(N)$. Thus, $\phi(N)$ is normal in $\phi(G)$.
2. Let a be an arbitrary element of G, then the set $a\phi^{-1}(N')a^{-1}$ satisfies that $\phi(a\phi^{-1}(N')a^{-1}) = \phi(a)\phi(\phi^{-1}(N')\phi(a^{-1}) \subseteq \phi(a)N'\phi(a)^{-1} \subseteq N'$ since N' is normal in G'. Thus, $a\phi^{-1}(N')a \subseteq \phi^{-1}(N')$ for every $a\ \in\ G$. This shows that $\phi^{-1}(N')$ is a normal subgroup of G.

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Corollary 3.13. [27],[28] The alternating group A_n is a subgroup of the symmetric group S_n.

Theorem 3.14. Let $\Gamma(S_n)$, $\Gamma(D_n)$ and $\Gamma(A_n)$ be the subgroups graphs of symmetric groups, S_n, dihedral groups D_n and the alternating groups A_n, $n \geq 3$. Suppose K and M are vertices on $\Gamma(D_n)$ and $\Gamma(A_n)$ respectively, then both K and M are also vertices on $\Gamma(S_n)$ and are adjacent to S_n if and only if K is adjacent to D_n on $\Gamma(D_n)$ and M is adjacent to A_n on $\Gamma(A_n)$.

Proof: Let $\Gamma(S_n)$, $\Gamma(D_n)$ and $\Gamma(A_n)$ be the subgroups graphs of symmetric groups, S_n, dihedral groups D_n and the alternating groups A_n, $n \geq 3$. To show that K and M which are vertices on $\Gamma(D_n)$ and $\Gamma(A_n)$ respectively are also vertices on $\Gamma(S_n)$ and are adjacent to S_n if and only if K is adjacent to D_n on $\Gamma(D_n)$ and M is adjacent to A_n on $\Gamma(A_n)$, then it suffices, if we can show both K and M to be subgroups of S_n.

Assume that K is a subgroup of D_n and M is a subgroup of A_n. Then observe the structure of the group of symmetries of a regular $n-$gon in a plane (dihedral group (D_n)), it is isomorphic to a subgroup of S_n then it is a proper subgroup of S_n. Also, by Corollary 3.13, A_n is a subgroup of S_n. Moreover, since $K \leq D_n$and $M \leq A_n$ by implication, they are also subgroups of S_n and hence are vertices on $\Gamma(S_n)$.

Conversely, assume that K is adjacent to D_n on $\Gamma(D_n)$ and M is adjacent to A_n on $\Gamma(A_n)$, then by Definition 3.1, $K \leq D_n$ and $M \leq A_n$. But both D_n and A_n are subgroups of S_n, also, by implication, they are adjacent to S_n.

Theorem 3.15. [29] If $n \geq 3$, then the number of subgroups of the dihederal group D_n is $\tau\ (n) + \sigma(n)$. Where $\tau\ (n)$ is the number of divisors of n and $\sigma(n)$ is the sum of divisors of n.

Remark 3.16. Let D_n be a dihedral group of order $n \geq 3$, then the number of vertices on the subgroups graphs $\Gamma(D_n)$ of the dihederal group D_n is $\tau(n) + \sigma(n)$, where $\tau(n)$ is the number of divisors of n and $\sigma(n)$ is the sum of divisors of n.

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Theorem 3.17. Let C be a commutator subgroup of a finite group G of order n, suppose there exist a normal subgroup N of G such that G/N is Abelian, then C and N are adjacent on the subgroup graphs of G.

Proof: Let G be a finite group of order n and C the commutator subgroup of G. If there exist a normal subgroup N of G such that G/N is Abelian, then to show that C and N are adjacent on $\Gamma(G)$ (the subgroup graph of G), we must show that either $C \leq N$ or $N \leq C$. Note that N is normal in G and G/N is Abelian, then for $x, y \in G$, we have $(xN)(yN) = (yN)(xN)$ and using the definition of Coset multiplication $xyN = yxN$. Which implies $xy(yx)^{-1} \in N$, where $xy(yx)^{-1} = xyx^{-1}y^{-1}$. Similarly, $xyx^{-1}y^{-1} \in N$ and since x and y are arbitrary then any commutator in G is an element of N and since N is a subgroup of G then any finite product of commutators in G is an element of N and thus $C \leq N$.

Theorem 3.18. Let C be a commutator subgroup of a finite group G of order n and let H be a subgroup of G. If there exist an $x \in G$ such that $[x, H] \subseteq C_G(H)$, then $[x, H]$ and C are adjacent on the subgroup graphs of G.

Proof: By Lemma 1.15(2) and using the method of [25], the map sending $h \in H$ to $[x, H^{-1}]$ is a homomorphism. Thus, the image map of H is the subgroup $[x, H]$.

Theorem 3.19. (Schur Zassenhaus) [16] Let G be a finite group and write $|G| = ab$ where $(a, b) = 1$. If G has a normal subgroup of order a then it has a subgroup of order b.

Remark 3.20. Let G be a finite group and write $|G| = ab$ where $(a, b) = 1$. Then the vertex set $V\ (\Gamma(G))$ of the subgroup graph of G contains at least two vertices $H, K \leq G$ which orders are a and b respectively.

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Theorem 3.21. Let G be a finite nilpotent group, such that G' is an abelian p-group with the minimal number of elements needed to generate G'; $d(G') \leq 3$, then G' is a vertex on the subgroup graph $\Gamma(G)$ and it is adjacent to G.

Proof: Suppose G is a finite nilpotent group, such that G' is an abelian p-group with $d(G') \leq 3$, then to show that G' is a vertex adjacent to G on the subgroup graph $\Gamma(G)$ of G, it will suffice if we can show G' to be equal to C, the commutator subgroup of G. Now, since G is finite, we assume an arbitrary $P \in Syl_P(G)$ such that $G' \leq P$ and obviously, P is normal in G and by Theorem 3.19 (Schur Zassenhaus theorem) and the methods in [25], $G = PQ$ where $P \cap Q = \{e\}$. Also, by Lemma 1.16(3), and Lemma 1.17, we set $R = [T, P]$ and if $G' = R$ there exist $t_1, t_2 \in T$ such that $R = [t_1, R][t_2, R]$ and $G' = \{[t_1 a, t_2 b] \mid a, b \in R\} = C$. Thus, we can assume $d(R) \leq 2$, and they also exists $t \in T$ with $R = [t, P]$. Since G/R is nilpotent, by Theorem 1.18 and Lemma 1.19,

$$G' = R(C) = R\left(\bigcup_{s \in P} \Gamma_s(P)\right) = \bigcup_{s \in P} \Gamma_{ts}(P) = C$$

Lemma 3.22. Let G and H be two non nilpotent finite groups, such that there is an isomorphism ϕ of G' and H', if the commutator subgroup C of G is adjacent to G on the subgroups graph $\Gamma(G)$ of G, then the commutator subgroup C of H is also adjacent to H on the subgroups graph $\Gamma(H)$ of H.

Proof: Suppose G and H are non nilpotent finite groups such that there is an isomorphic map between G' and H', then we can safely say there is also isomorphic map between the commutator subgroups of G' and H', which shows the isomorphic relationship between G and H. Also, since the commutator subgroup of G is adjacent to G on $\Gamma(G)$ then the commutator subgroup of H is also adjacent to H on $\Gamma(H)$.

Example 3.23. Let Q_8 be a quaternion group generated by the following matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

[30], using the matrix multiplication obtained $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, observe that the subgroups of Q_8 consists of Q_8 itself and of the cyclic subgroups $\langle 1 \rangle = \{e\}, \langle -1 \rangle = \{1, -1\}, \langle i \rangle = \{1, i, -1, -i\}, \langle j \rangle = \{1, j, -1, -j\}, \langle k \rangle = \{1, k, -1, -k\}$ and the following is the subgroups graph of Q_8. See Figure 2.

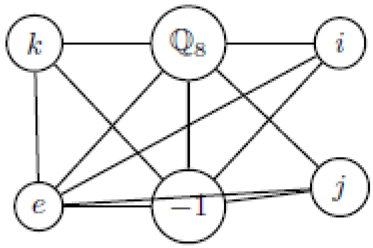

Figure 2. Subgroups graph of Q8.

4 Conclusion

This study has highlighted some algebraic properties and combinatorial structures of Subgroups graph $\Gamma(G)$ of finite groups. The connections between the Subgroups graphs of finite groups upto homomorphism and isomorphism were also studied and further looked at the relationships between the subgroups graphs of symmetric groups, S_n, dihedral groups D_n and the alternating groups An, when $n \geq 3$.

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