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Expected Value of the Occupation Times of Brownian Motion

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Abstract

The duration spent by a stochastic process within a specific spatial range over a given finite time period is referred to as the occupation times of the process. In the nonwoven production industry, this phenomenon manifests itself during the fiber lay-down process. The occupation time can be understood as the mass of fiber material accumulated within a specific area. It is crucial to have knowledge of the average mass per unit area of the final fleece from an application standpoint. In this paper derive an expression for the expected value of the occupation times in terms of Gaussian error functions.

Keywords: occupation times, Brownian motion, expected value

1 Introduction

Technical textiles have garnered significant interest across various industries in recent decades because of their cost-effective production methods. Random fiber webs are created by overlapping numerous individual slender fibers, resulting in nonwoven materials that are utilized in various industries such as textile, construction, and hygiene. These materials serve as essential components in products like baby diapers, clothing textiles, filters, and medical devices, among others. Endless fibers are manufactured through melt-spinning processes, where numerous individual fibers are created by continuously extruding molten polymer through closely spaced narrow nozzles arranged in a row on a spinning beam. The viscous or viscoelastic fibers undergo stretching and spinning processes until they solidify as a result of exposure to cooling air currents. Prior to being placed on a moving conveyor belt to create a web, the elastic fibers tangle and form loops because of the turbulent air flows. The homogeneity and load capacity of the fiber web are crucial textile characteristics when evaluating the quality of industrial nonwoven fabrics. The modeling and simulation of fiber dynamics and lay-down are essential for optimizing and controlling fleece quality. The fleece's mass per unit area is typically the available data used to assess its quality, particularly when evaluating it on an industrial scale.

Due to its complexity it is not possible to study the whole process using mathematical means at a stroke. In recent years, research has led to the development of a series of models that effectively capture specific elements of the process chain. The papers [4, 6, 9] presented and examined a probabilistic model concerning the fiber deposition process in nonwoven manufacturing. The model relies on stochastic differential equations to depict the final position of the fiber on the conveyor belt, taking into account the impact of turbulent air currents. In [1] the estimation of the Ornstein-Uhlenbeck process's parameter from the available data on mass per unit area, the occupation time in mathematical terms, was done.

Definition 1.1. *Let $[a,b] \subset \mathbb{R}$ be a compact interval in $\mathbb{R}$ and let $X = (X_t)_{t\in[0,T]}$, $0 < T < \infty$, be a stochastic process. The occupation time $M_{T,[a,b]}(X)$ is defined as*

$$M_{T,[a,b]}(X) := \int_0^T \mathbf{1}_{[a,b]}(X_t)\, dt = \int_0^T \int_a^b \delta_0\left(X_t - x\right)\, dx\, dt,$$

where δ_0 and $\mathbf{1}_{[a,b]}$ is the Dirac-delta function and the indicator function of the interval $[a,b]$, respectively.

Occupation times formally represent the duration that a stochastic process occupies the spatial interval $[a,b]$ within the time interval $[0,T]$. In terms of our physical framework for the manufacturing of nonwoven materials, the occupation time can be understood as the quantity of fiber material accumulated within the range $[a,b]$, specifically the mass per unit area of the resulting fleece.

In this research, we investigate the occupation time of one-dimensional Brownian motion based on our previous work [18]. In that paper we show that occupation times of one-dimensional Brownian motion is a Hida distribution. The present paper derives the explicit form of the expectation of the occupation times in terms of the Gaussian error function. In subsequent investigations, we employ a white noise approach to extend the concept to higher dimensions, despite the availability of classical probabilistic methods for studying the problem. In Section 2, we present essential background information on the theory of white noise. The main result, along with its proof, is presented in Section 3.

2 White Noise Analysis

This section provides essential background information on the theory of white noise. For a more thorough examination of white noise theory we refer to [8, 11] and references therein. We start with the Gelfand triple

$$\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}'(\mathbb{R}),$$

where $L^2(\mathbb{R})$ is the real Hilbert space of all real-valued Lebesgue square-integrable functions, $\mathcal{S}(\mathbb{R})$ is the space of real-valued Schwartz test function, and $\mathcal{S}'(\mathbb{R})$ is the space of real-valued tempered distributions. We construct a probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{C}, \mu)$ where $\mathcal{C}$ is the Borel σ-algebra on $\mathcal{S}'(\mathbb{R})$ generated by cylinder sets. The Bochner-Minlos theorem guarantees the unique determination of the probability measure μ by specifying the characteristic function

$$C(f) := \int_{\mathcal{S}'(\mathbb{R})} \exp\left(i\langle \omega, f\rangle\right)\, d\mu(\omega) = \exp\left(-\frac{1}{2}|f|_0^2\right)$$

for all $f \in \mathcal{S}(\mathbb{R})$. The standard norm in the space $L^2(\mathbb{R})$ is denoted by $|\cdot|_0$, while the dual pairing between the spaces $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ is denoted by $\langle \cdot, \cdot \rangle$. The dual pairing is regarded as the extension of the inner product on $L^2(\mathbb{R})$, i.e.

$$\langle g, f\rangle = \int_{\mathbb{R}} g(x) f(x)\, dx,$$

for all $g \in L^2(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. The real-valued white noise space is identified by this probability space as it encompasses the paths of the one-dimensional Gaussian white noise. In this framework a one-dimensional Brownian motion can be defined by a continuous modification of the process $B = (B_t)_{t\geq 0}$ with

$$B(t) := \left\langle \cdot, \mathbf{1}_{[0,t]} \right\rangle,$$

where $\mathbf{1}$ denotes the indicator function.

In the sequel, we shall employ the the Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) := L^2\left(\mathcal{S}'(\mathbb{R}), \mathcal{C}, \mu\right) \hookrightarrow (\mathcal{S})^*$$

where $(\mathcal{S})$ is the space of white noise test functions and $(\mathcal{S})^*$ is the topological dual space of $(\mathcal{S})$. The terms *Hida test functions* and *Hida distributions* refer to the components of $(\mathcal{S})$ and $(\mathcal{S})^*$, respectively. In this setting, white noise can be precisely characterized as the temporal rate of change of Brownian motion with respect to the topology of $(\mathcal{S})^*$. The S-transform, a crucial tool in the white noise analysis, is defined for any $\Phi \in (\mathcal{S})^*$ as

$$(\mathrm{S}\Phi)(\varphi) := \langle\langle \Phi, :\exp(\langle \cdot, \varphi\rangle):\rangle\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

Here,

$$:\exp(\langle \cdot, \varphi\rangle) ::= C(\varphi)\exp(\langle \cdot, \varphi\rangle),$$

is the so-called Wick exponential and $\langle\langle \cdot, \cdot\rangle\rangle$ denotes the topological dual pairing between $(\mathcal{S})^*$ and $(\mathcal{S})$. The S-transform may be considered as the analog of the Laplace transform in an infinite dimensional space with respect to the Gaussian measure. The S-transform offers a convenient method for identifying a Hida distribution $\Phi \in (\mathcal{S})^*$, especially in cases where finding the explicit form of the Wiener-Itô chaos decomposition of Φ proves to be challenging..

Theorem 2.1. *[11] The S-transform is injective, i.e. if $S\Phi(\varphi) = S\Psi(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R})$, then $\Phi = \Psi$.*

In the subsequent discussion, we present a condition that is sufficient for the Bochner integrability of a collection of Hida distributions that are contingent upon an additional parameter.

Theorem 2.2. *[10] Let $(\Omega, \mathcal{A}, \nu)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ be a function from Ω to $(\mathcal{S})^*$. If*

(1) the function $\lambda \mapsto \mathrm{S}(\Phi_\lambda)(\varphi)$ is measurable for all $\varphi \in \mathcal{S}(\mathbb{R})$, and

(2) there are $C_1(\lambda) \in L^1(\Omega, \mathcal{A}, \nu)$, $C_2(\lambda) \in L^\infty(\Omega, \mathcal{A}, \nu)$ and a continuous seminorm $\|\cdot\|$ on $\mathcal{S}(\mathbb{R})$ such that for all $z \in \mathbb{C}$, $\varphi \in \mathcal{S}(\mathbb{R})$

$$|\mathrm{S}(\Phi_\lambda)(z\varphi)| \leq C_1(\lambda)\exp\left(C_2(\lambda)|z|^2\|\varphi\|^2\right),$$

then Φ_λ is Bochner integrable in $(\mathcal{S})^*$. It follows that $\int_\Omega \Phi_\lambda \, d\nu(\lambda) \in (\mathcal{S})^*$. Moreover,

$$\mathrm{S}\left(\int_\Omega \Phi_\lambda \, d\nu(\lambda)\right) = \int_\Omega \mathrm{S}(\Phi_\lambda) \, d\nu(\lambda).$$

We define Donsker's delta distribution by

$$\delta_0\left(B_t - x\right) = \delta_0\left(\left\langle \cdot, \mathbf{1}_{[0,t]}\right\rangle - x\right) := \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\lambda\left(\left\langle \cdot, \mathbf{1}_{[0,t]}\right\rangle - x\right)\right) \, d\lambda.$$

It has been demonstrated that $\delta_0\left(B_t - x\right) \in (\mathcal{S})^*$. Moreover, the S-transform of the aforementioned is expressed as

$$S\delta_0\left(B_t - x\right)(\varphi) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(\left\langle \varphi, \mathbf{1}_{[0,t]}\right\rangle - x\right)^2\right),$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$. For further information and evidence, refer to the sources such as [8, 11]. The Donsker delta distribution holds significant importance as a subject of study in the realm of Gaussian analysis. For instance, it can be employed to examine local times, self-intersection local times, stochastic current, and Feynman integrals, see e.g. [2, 3, 7, 13, 14, 17]. The derivatives of Donsker's delta distribution has been also investigaed in previous research such as [15]. In the context of stochastic processes with memory, the analysis of Donsker's delta distribution is presented in [16].

3 Main Result

In [18] the following results have been proved.

Theorem 3.1. *1. Let $B = (B_t)_{t\in[0,T]}$ be a one-dimensional standard Brownian motion and let $[a, b] \subset \mathbb{R}$ be a compact interval. The occupation time*

$$M_{T,[a,b]}(B) := \int_0^T \int_a^b \delta_0\left(B_t - x\right) \, dx \, dt$$

is a Hida distribution.

2. The expression for the S-transform of the occupation times of Brownian motion, for any $\varphi \in \mathcal{S}(\mathbb{R})$, is provided by

$$SM_{T,[a,b]}(B)(\varphi) = \int_0^T \frac{1}{\sqrt{2\pi t}} \int_a^b \exp\left(-\frac{1}{2t}\left(\int_0^t \varphi(s)\, ds - x\right)^2\right) dx \, dt.$$

.

In this paper we improve the above result by deriving an explicit form for the expected value of the occupation times of Brownian motion in terms of the Gaussian error function:

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy, \quad x > 0.$$

The following integration formula are required for this purpose.

Lemma 3.2 ([12], formula 14)**.** *For any* $n \geq 2$

$$\int \mathrm{erf}(az) z^{-n}\, dz = -\frac{\mathrm{erf}(az)}{(n-1)z^{n-1}} + \frac{2a}{(n-1)\sqrt{\pi}} \int \frac{1}{z^{n-1}} e^{-a^2 z^2}\, dz.$$

Lemma 3.3 ([5], formula 3.461(5))**.** *For any* $u > 0$ *and* $|\mathrm{arg}\mu| < \frac{\pi}{2}$

$$\int_u^\infty \frac{e^{-\mu x^2}}{x^2}\, dx = \frac{1}{u} e^{\mu x^2} - \sqrt{\mu\pi}\left(1 - \mathrm{erf}\left(u\sqrt{\mu}\right)\right).$$

Now, we are ready to prove our main result.

Theorem 3.4. *The expectation of the occupation times* $M_{T,[a,b]}(B)$ *of one-dimensional standard Brownian motion* $B = (B_t)_{t\in[0,T]}$ *is given by*

$$\begin{aligned}&\mathbb{E}_\mu\left(M_{T,[a,b]}(B)\right)\\ &= \sqrt{\frac{T}{2\pi}}\left(be^{-\frac{b^2}{2T}} - ae^{-\frac{a^2}{2T}}\right) - \frac{b^2-a^2}{2} + \frac{T+b^2}{2}\mathrm{erf}\left(\frac{b}{\sqrt{2T}}\right) - \frac{T+a^2}{2}\mathrm{erf}\left(\frac{a}{\sqrt{2T}}\right).\end{aligned}$$

Proof: Since $M_{T,[a,b]}(B) = \int_0^T \int_a^b \delta_0\left(B_t - x\right)\, dx\, dt \in (\mathcal{S})^*$, then, using Theorem 2.2, we have

$$\begin{aligned}\mathbb{E}_\mu\left(M_{T,[a,b]}(B)\right) &= SM_{T,[a,b]}(B)(0)\\ &= S\left(\int_0^T \int_a^b \delta_0\left(B_t - x\right)\, dx\, dt\right)(0)\\ &= \int_0^T \int_a^b S\delta_0\left(B_t - x\right)(0)\, dx\, dt\\ &= \int_0^T \int_a^b \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}\left(\langle 0, \mathbf{1}_{[0,t]}\rangle - x\right)^2}\, dx\, dt\\ &= \int_0^T \int_a^b \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx\, dt\\ &= \int_0^T \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}}\, dx\, dt\\ &= \int_0^T \frac{1}{\sqrt{2\pi t}} \int_{a/\sqrt{2t}}^{b/\sqrt{2t}} \sqrt{2t} e^{-y^2}\, dy\, dt\end{aligned}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^T \left(\int_0^{b/\sqrt{2t}} \sqrt{2t} e^{-y^2}\, dy - \int_0^{a/\sqrt{2t}} \sqrt{2t} e^{-y^2}\, dy \right) dt$$
$$= \frac{1}{2} \int_0^T \left(\mathrm{erf}\left(\frac{b}{\sqrt{2t}}\right) - \mathrm{erf}\left(\frac{a}{\sqrt{2t}}\right) \right) dt.$$

Let us now compute $\int_0^T \mathrm{erf}\left(\frac{b}{\sqrt{2t}}\right) dt$. Using substitution $r = \frac{b}{\sqrt{2t}}$ we get

$$\int_0^T \mathrm{erf}\left(\frac{b}{\sqrt{2t}}\right) dt = b^2 \int_{b/\sqrt{2T}}^{\infty} \frac{\mathrm{erf}(r)}{r^3}\, dr.$$

An application of Lemma 3.2 yields

$$\int \frac{\mathrm{erf}(r)}{r^3}\, dr = -\frac{\mathrm{erf}(r)}{2r^2} + \frac{1}{\sqrt{\pi}} \int \frac{1}{r^2} e^{-r^2}\, dr.$$

Hence,

$$\int_{b/\sqrt{2T}}^{\infty} \frac{\mathrm{erf}(r)}{r^3}\, dr = \lim_{s\to\infty} \left[-\frac{\mathrm{erf}(r)}{2r^2} \right]_{b/\sqrt{2T}}^{s} + \frac{1}{\sqrt{\pi}} \int \frac{1}{r^2} e^{-r^2}\, dr$$
$$= \frac{T \mathrm{erf}\left(\frac{b}{\sqrt{2T}}\right)}{b^2} + \frac{1}{\sqrt{\pi}} \int_{b/\sqrt{2T}}^{\infty} \frac{1}{r^2} e^{-r^2}\, dr.$$

Since $\frac{b}{\sqrt{2T}} > 0$ and $|\arg 1| < \frac{\pi}{2}$, then, by using Lemma 3.3, we obtain

$$\int_{b/\sqrt{2T}}^{\infty} \frac{1}{r^2} e^{-r^2}\, dr = \frac{\sqrt{2T}}{b} e^{-\frac{b^2}{2T}} - \sqrt{\pi} \left(1 - \mathrm{erf}\left(\frac{b}{\sqrt{2T}}\right) \right).$$

Therefore,

$$\int_{b/\sqrt{2T}}^{\infty} \frac{\mathrm{erf}(r)}{r^3}\, dr = \frac{T \mathrm{erf}\left(\frac{b}{\sqrt{2T}}\right)}{b^2} + \frac{1}{b}\sqrt{\frac{2T}{\pi}} e^{-\frac{b^2}{2T}} - \left(1 - \mathrm{erf}\left(\frac{b}{\sqrt{2T}}\right) \right)$$

and

$$\int_0^T \mathrm{erf}\left(\frac{b}{\sqrt{2t}}\right) dt = b^2 \int_{b/\sqrt{2T}}^{\infty} \frac{\mathrm{erf}(r)}{r^3}\, dr$$
$$= b^2 \left(\frac{T \mathrm{erf}\left(\frac{b}{\sqrt{2T}}\right)}{b^2} + \frac{1}{b}\sqrt{\frac{2T}{\pi}} e^{-\frac{b^2}{2T}} - \left(1 - \mathrm{erf}\left(\frac{b}{\sqrt{2T}}\right) \right) \right)$$
$$= (T + b^2)\mathrm{erf}\left(\frac{b}{\sqrt{2T}}\right) + \sqrt{\frac{2T}{\pi}} b e^{-\frac{b^2}{2T}} - b^2.$$

Similarly,

$$\int_0^T \mathrm{erf}\left(\frac{a}{\sqrt{2t}}\right) dt = (T + a^2)\mathrm{erf}\left(\frac{a}{\sqrt{2T}}\right) + \sqrt{\frac{2T}{\pi}} a e^{-\frac{a^2}{2T}} - a^2.$$

Finally, we have

$$
\begin{aligned}
&\mathbb{E}_\mu\left(M_{T,[a,b]}(B)\right)\\
&=\frac{1}{2}\int_0^T\left(\operatorname{erf}\left(\frac{b}{\sqrt{2t}}\right)-\operatorname{erf}\left(\frac{a}{\sqrt{2t}}\right)\right)dt\\
&=\frac{1}{2}\Bigg((T+b^2)\operatorname{erf}\left(\frac{b}{\sqrt{2T}}\right)+\sqrt{\frac{2T}{\pi}}be^{-\frac{b^2}{2T}}-b^2-(T+a^2)\operatorname{erf}\left(\frac{a}{\sqrt{2T}}\right)\\
&\qquad-\sqrt{\frac{2T}{\pi}}ae^{-\frac{a^2}{2T}}+a^2\Bigg)\\
&=\sqrt{\frac{T}{2\pi}}\left(be^{-\frac{b^2}{2T}}-ae^{-\frac{a^2}{2T}}\right)-\frac{b^2-a^2}{2}+\frac{T+b^2}{2}\operatorname{erf}\left(\frac{b}{\sqrt{2T}}\right)-\frac{T+a^2}{2}\operatorname{erf}\left(\frac{a}{\sqrt{2T}}\right).
\end{aligned}
$$

The proof is finished. ■

4 Conclusions

In this paper we derive an explicit expression for the expected value of the occupation times of a standard one-dimensional Brownian motion in terms of Gaussian error function. This explicit form is preferred from an application standpoint as it offers greater utility for computational tasks. Our current findings are constrained to a one-dimensional setting. As a further research plan, we will generalize the present result to higher spatial dimensions.

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