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A White Noise Approach to Occupation Times of Brownian Motion

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Abstract

Occupation times of a stochastic process models the amount of time the process spends inside a spatial interval during a certain finite time horizon. It appears in the fiber lay-down process in nonwoven production industry. The occupation time can be interpreted as the mass of fiber material deposited inside some region. From application point of view, it is important to know the average mass per unit area of the final fleece. In this paper we use white noise theory to prove the existence of the occupation times of one-dimensional Brownian motion and provide an expression for the expected value of the occupation times.

Keywords: occupation times, Brownian motion, white noise analysis

1 Introduction

Technical textiles have attracted great attention to diverse branches of industry over the last decades due to their comparatively cheap manufacturing. By overlapping thousands of individual slender fibers, random fiber webs emerge yielding nonwoven materials that find applications e.g. in textile, building and hygiene industry as integral components of baby diapers, closing textiles, filters and medical devices, to name but a few. They are

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produced in melt-spinning operations: hundreds of individual endless fibers are obtained by the continuous extrusion of a molten polymer through narrow nozzles that are densely and equidistantly placed in a row at a spinning beam. The viscous or viscoelastic fibers are streched and spun until they solidify due to cooling air streams. Before the elastic fibers lay down on a moving conveyor belt to form a web, they become entangled and form loops due to the highly turbulent air flows. The homogeneity and load capacity of the fiber web are the most important textile properties for quality assesment of industrial nonwoven fabrics. The optimization and control of the fleece quality require modeling and simulation of fiber dynamics and lay-down. Available data to judge the quality, at least on the industrial scale, are usually the mass per unit area of the fleece.

Since the mathematical treatment of the whole process at a stroke is not possible due to its complexity, a hierarchy of models that adequately describe partial aspects of the process chain has been developed in research during the last years. A stochastic model for the fiber deposition in the nonwoven production was proposed and analyzed in [4, 5, 7, 10]. The model is based on stochastic differential equations describing the resulting position of the fiber on the belt under the influence of turbulent air flows. In [1] parameter estimation of the Ornstein-Uhlenbeck process from available mass per unit area data, the occupation time in mathematical terms, was done.

Definition 1.1. *Let $X = (X_t)_{t\in[0,T]}$, $T > 0$, be a stochastic process and consider an interval $[a, b] \subset \mathbb{R}$. The occupation time $M_{T,[a,b]}(X)$ is defined as*

$$M_{T,[a,b]}(X) := \int_0^T \mathbf{1}_{[a,b]}(X_t)\, dt = \int_0^T \int_a^b \delta_0 \left(X_t - x\right)\, dx\, dt.$$

Here, $\mathbf{1}_{[a,b]}$ denoted the indicator function of the interval $[a, b]$ and δ_0 is the Dirac-delta distribution.

Formally, occupation times models the time the stochastic process spends inside the spatial interval [a, b] during the time interval $[0, T]$. In terms of our physical model for the nonwoven production, the occupation time can be interpreted as the mass of fiber material deposited inside the interval $[a, b]$, i.e. the mass per unit area of the final fleece.

Motivated by the above mentioned problem, in this paper we study the occupation time of one-dimensional Brownian motion. In particular, we show that occupation times of one-dimensional Brownian motion is a white noise distribution in the sense of Hida.

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Although it is possible to study the problem by classical probabilistic method, we use a white noise approach to generalize the concept also to higher dimensions in later research. Moreover, in future work an extension to more general process (e.g. with fractional Gaussian noise) is planed. In the next section we provide neccesary background on the white noise theory. The main result together with its proof are given afterward.

2 White Noise Analysis

In this section we give background on the white noise theory used throughout this paper. For a more comprehensive discussions including various applications of white noise theory we refer to [8, 9, 12, 13] and references therein. We start with the Gelfand triple

$$\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}'(\mathbb{R}),$$

where $\mathcal{S}(\mathbb{R})$ is the space of real-valued Schwartz test function, $\mathcal{S}'(\mathbb{R})$ is the space of real-valued tempered distributions, and $L^2(\mathbb{R})$ is the real Hilbert space of all real-valued Lebesgue square-integrable functions. Next, we construct a probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{C}, \mu)$ where $\mathcal{C}$ is the Borel σ-algebra generated by cylinder sets on $\mathcal{S}'(\mathbb{R})$ and the unique probability measure μ is established through the Bochner-Minlos theorem by fixing the characteristic function

$$C(f) := \int_{\mathcal{S}'(\mathbb{R})} \exp\left(i\langle \omega, f\rangle\right)\, d\mu(\omega) = \exp\left(-\frac{1}{2}|f|_0^2\right)$$

for all $f \in \mathcal{S}(\mathbb{R})$. Here $|\cdot|_0$ denotes the usual norm in the $L^2(\mathbb{R})$, and $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. The dual pairing is considered as the bilinear extension of the inner product on $L^2(\mathbb{R})$, i.e.

$$\langle g, f\rangle = \int_{\mathbb{R}} g(x) f(x)\, dx,$$

for all $g \in L^2(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. This probability space is known as the real-valued white noise space since it contains the sample paths of the one-dimensional Gaussian white noise. In this setting a one-dimensional Brownian motion can be represented by a continuous modification of the stochastic process $B = (B_t)_{t\geq 0}$ with

$$B(t) := \left\langle \cdot, \mathbf{1}_{[0,t]} \right\rangle,$$

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where $\mathbf{1}_A$ denotes the indicator function of a set $A \subset \mathbb{R}$.

In the sequel we will use the Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) := L^2\left(\mathcal{S}'(\mathbb{R}), \mathcal{C}, \mu\right) \hookrightarrow (\mathcal{S})^*$$

where $(\mathcal{S})$ is the space of white noise test functions obtained by taking the intersection of a family of Hilbert subspaces of $L^2(\mu)$. The space of white noise distributions $(\mathcal{S})^*$ is defined as the topological dual space of $(\mathcal{S})$. Elements of $(\mathcal{S})$ and $(\mathcal{S})^*$ are known as *Hida test functions* and *Hida distributions*, respectively. Within this framework white noise can be considered as the time derivative of Brownian motion with respect to the topology of $(\mathcal{S})^*$. An important tool in white noise analysis is the S-transform which can be considered as the Laplace transform with respect to the infinite dimensional Gaussian measure. The S-transform of $\Phi \in (\mathcal{S})^*$ is defined as

$$(\mathrm{S}\Phi)(\varphi) := \langle\langle \Phi, : \exp\left(\langle \cdot, \varphi\rangle\right) : \rangle\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}),$$

where

$$: \exp\left(\langle \cdot, \varphi\rangle\right) ::= C(\varphi) \exp\left(\langle \cdot, \varphi\rangle\right),$$

is the so-called Wick exponential and $\langle\langle \cdot, \cdot\rangle\rangle$ denotes the dual pairing between $(\mathcal{S})^*$ and $(\mathcal{S})$. We define this dual pairing as the bilinear extension of the sesquilinear inner product on $L^2(\mu)$. The S-transform provides a convenient way to identify a Hida distribution $\Phi \in (\mathcal{S})^*$, in particular, when it is hard to find the explicit form for the Wiener-Itô chaos decomposition of Φ.

Theorem 2.1. *[12]*

1. *The S-transform is injective, i.e. if $S\Phi(\varphi) = S\Psi(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R})$, then $\Phi = \Psi$.*
2. *If a stochastic distribution process $X : I \to (\mathcal{S})^*$ is differentiable, then $S\frac{d}{dt}X(t)(\varphi) = \frac{d}{dt}SX(t)(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R})$.*

In the following we state a sufficient condition on the Bochner integrability of a family of Hida distributions which depend on an additional parameter.

Theorem 2.2. *[11] Let $(\Omega, \mathcal{A}, \nu)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ be a mapping from Ω to $(\mathcal{S})^*$. If*

(1) the mapping $\lambda \mapsto \mathrm{S}(\Phi_\lambda)(\varphi)$ *is measurable for all* $\varphi \in \mathcal{S}(\mathbb{R})$*, and*

(2) there exist $C_1(\lambda) \in L^1(\Omega, \mathcal{A}, \nu)$*,* $C_2(\lambda) \in L^\infty(\Omega, \mathcal{A}, \nu)$ *and a continuous semi-norm* $\|\cdot\|$ *on* $\mathcal{S}(\mathbb{R})$ *such that for all* $z \in \mathbb{C}$*,* $\varphi \in \mathcal{S}(\mathbb{R})$

$$|\mathrm{S}(\Phi_\lambda)(z\varphi)| \leq C_1(\lambda) \exp\left(C_2(\lambda)|z|^2 \|\varphi\|^2\right),$$

then Φ_λ *is Bochner integrable with respect to some Hilbertian norm which topologizing* $(\mathcal{S})^*$*. Hence* $\int_\Omega \Phi_\lambda \, d\nu(\lambda) \in (\mathcal{S})^*$*, and furthermore*

$$\mathrm{S}\left(\int_\Omega \Phi_\lambda \, d\nu(\lambda)\right) = \int_\Omega \mathrm{S}(\Phi_\lambda) \, d\nu(\lambda).$$

Let $0 < T < \infty$ and $B = (B_t)_{t\in[0,T]}$ be a one-dimensional standard Brownian motion. The corresponding Donsker's delta distribution is given by

$$\delta_0(B_t - x) = \delta_0\left(\langle \cdot, \mathbf{1}_{[0,t]}\rangle - x\right) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\lambda\left(\langle \cdot, \mathbf{1}_{[0,t]}\rangle - x\right)\right) \, d\lambda.$$

It has been proved that $\delta_0(B_t - x) \in (\mathcal{S})^*$. Furthermore, its S-transform is given by

$$S\delta_0(B_t - x)(\varphi) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(\langle \varphi, \mathbf{1}_{[0,t]}\rangle - x\right)^2\right),$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$. For details and proofs see e.g. [8, 12]. The Donsker delta distribution is an important research object in the Gaussian analysis. For example, it can be used to study local times, self-intersection local times, stochastic current and Feynman integrals, see e.g. [2, 3, 6, 14, 15, 18]. The derivatives of Donsker's delta distribution has been also studied in [16]. In [17] Donsker's delta distribution is analyzed in the context of stochastic processes with memory.

3 Main Result

Now we are ready to prove the main finding of the paper.

Theorem 3.1. *Let* $B = (B_t)_{t\in[0,T]}$ *be a one-dimensional standard Brownian motion and consider an interval* $[a, b] \subset \mathbb{R}$*. The occupation time*

$$M_{T,[a,b]}(B) := \int_0^T \int_a^b \delta_0(B_t - x) \, dx \, dt$$

is a Hida distribution.

Proof: It is apparent, at least formally, that occupation times can be obtained by integrating Donsker's delta distribution with respect to the product measure on $[a,b]\times[0,T]$. In this regard we will use Kondratiev-Streit integration theorem (Theorem 2.2) to prove the statement. Observe that for any $\varphi\in\mathcal{S}(\mathbb{R})$

$$S\delta_0\left(B_t-x\right)(\varphi)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{1}{2t}\left(\int_0^t\varphi(s)\,ds-x\right)^2\right)$$

is a measurable function with respect to the product measure on $[0,T]\times[a,b]$. Now for any $z\in\mathbb{C}$ and $\varphi\in\mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}
&\left|S\delta_0\left(B_t-x\right)(z\varphi)\right|\\
&=\left|\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{1}{2t}\left(\int_0^t z\varphi(s)\,ds-x\right)^2\right)\right|\\
&=\frac{1}{\sqrt{2\pi t}}\left|\exp\left(-\frac{1}{2t}\left(\left\langle z\varphi,\mathbf{1}_{[0,t]}\right\rangle-x\right)^2\right)\right|\\
&=\frac{1}{\sqrt{2\pi t}}\left|\exp\left(-\frac{1}{2t}\left(\left\langle z\varphi,\mathbf{1}_{[0,t]}\right\rangle^2-2x\left\langle z\varphi,\mathbf{1}_{[0,t]}\right\rangle+x^2\right)\right)\right|\\
&=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\left|\exp\left(-\frac{1}{2t}\left\langle z\varphi,\mathbf{1}_{[0,t]}\right\rangle^2\right)\exp\left(\frac{x}{t}\left\langle z\varphi,\mathbf{1}_{[0,t]}\right\rangle\right)\right|\\
&\le\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\exp\left(\frac{1}{2t}\left|\left\langle z\varphi,\mathbf{1}_{[0,t]}\right\rangle\right|^2\right)\exp\left(\left|\frac{x}{t}\left\langle z\varphi,\mathbf{1}_{[0,t]}\right\rangle\right|\right)\\
&=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\exp\left(\frac{1}{2t}|z|^2\left|\left\langle\varphi,\mathbf{1}_{[0,t]}\right\rangle\right|^2\right)\exp\left(\frac{|x|}{\sqrt{t}}\frac{|z|}{\sqrt{t}}\left|\left\langle\varphi,\mathbf{1}_{[0,t]}\right\rangle\right|\right)\\
&\le\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\exp\left(\frac{1}{2t}|z|^2|\varphi|^2\left|\mathbf{1}_{[0,t]}\right|^2\right)\exp\left(\frac{x^2}{2t}\right)\exp\left(\frac{1}{2t}|z|^2|\varphi|^2\left|\mathbf{1}_{[0,t]}\right|^2\right)\\
&=\frac{1}{\sqrt{2\pi t}}\exp\left(|z|^2|\varphi|^2\right).
\end{aligned}$$

The first factor

$$C_1(t,x)=\frac{1}{\sqrt{2\pi t}}$$

is an integrable function on $[0,T]\times[a,b]$ while the second factor

$$C_2(t,x)=\exp\left(|z|^2|\varphi|^2\right)$$

is a bounded function of t and x. Theorem 2.2 gives the desired result. ■

Corollary 3.2. *The S-transform of the occupation times of Brownian motion is given by*

$$SM_{T,[a,b]}(B)(\varphi)=\int_0^T\frac{1}{\sqrt{2\pi t}}\int_a^b\exp\left(-\frac{1}{2t}\left(\int_0^t\varphi(s)\,ds-x\right)^2\right)dx\,dt,$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$.

Proof: Since $M_{T,[a,b]}(B) \in (\mathcal{S})^*$ then according to Theorem 2.2 for every $\varphi \in \mathcal{S}(\mathbb{R})$ it holds

$$\begin{aligned} SM_{T,[a,b]}(B)(\varphi) &= S\int_0^T \int_a^b \delta_0\left(B_t - x\right)\, dx\, dt(\varphi) \\ &= \int_0^T \int_a^b S\delta_0\left(B_t - x\right)(\varphi)\, dx\, dt \\ &= \int_0^T \int_a^b \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(\int_0^t \varphi(s)\, ds - x\right)^2\right) dx\, dt \\ &= \int_0^T \frac{1}{\sqrt{2\pi t}} \int_a^b \exp\left(-\frac{1}{2t}\left(\int_0^t \varphi(s)\, ds - x\right)^2\right) dx\, dt. \blacksquare \end{aligned}$$

Corollary 3.3. *The expected value of the occupation times of Brownian motion is given by*

$$\mathbb{E}_\mu\left(M_{T,[a,b]}(B)\right) = \int_0^T \frac{1}{\sqrt{2\pi t}} \int_a^b \exp\left(-\frac{x^2}{2t}\right) dx\, dt.$$

Proof: The expected value with respect to the white noise measure of the occupation times of Brownian motion is obtained by taking the S-transform and evaluating the value at 0:

$$\begin{aligned} \mathbb{E}_\mu\left(M_{T,[a,b]}(B)\right) &= SM_{T,[a,b]}(B)(0) \\ &= \int_0^T \frac{1}{\sqrt{2\pi t}} \int_a^b \exp\left(-\frac{1}{2t}\left(\int_0^t 0\, ds - x\right)^2\right) dx\, dt \\ &= \int_0^T \frac{1}{\sqrt{2\pi t}} \int_a^b \exp\left(-\frac{x^2}{2t}\right) dx\, dt. \blacksquare \end{aligned}$$

4 Conclusions

We give a mathematically rigorous meaning to the occupation times of a standard Brownian motion as a Hida distribution. An expression for the expected value for the occupation times is also obtained. For the application point of view it is desirable to have a more explicit form for the expected value. This will be done in the future work. We would like also to mention that our present result is limited to one-dimensional setting. For further research we plan to generalize the result to higher spatial dimensions.

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