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Identity Graph of Finite Cyclic Groups

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Abstract

This paper discusses how to express a finite group as a graph, specifically about the identity graph of a cyclic group. The term chosen for the graph is an identity graph, because it is the identity element of the group that holds the key in forming the identity graph. Through the identity graph, it can be seen which elements are inverse of themselves and other properties of the group. We will look for the characteristics of identity graph of the finite cyclic group, for both cases of odd and even order.

Keywords: Graph, identity graph, group, identity element.

1 Introduction

Mathematics as a science has several branches including abstract algebra and graph theory [1-3]. The phrase of abstract algebra has been used since the early 20th century to distinguish them from what is now more commonly referred to as elementary algebra, which is the study of the rules of manipulation of algebraic formulas and expressions involving real or complex variables and numbers [4-6]. Abstract algebra is a field of

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mathematics that studies algebraic structures, such as monoids, groups, rings, fields, modules, etc. [3, 4].

Students often find it difficult to learn abstract structure such as group. Therefore, some writers are looking for a way to represent a group with a diagram called a graph. Graph theory is a branch of mathematics that has been studied and developed by researchers. In its development, the application of graph theory is often found both in mathematics itself and in other fields such as computer science, biology, chemistry and in problems in human life such as transportation problems, installation of public facilities, and traffic light management.

In this paper, graph theory will be applied in abstract algebra, especially to represent groups in the form of a graph so that it can be visualized diagrammatically and studied its properties through the graph of the group. The group discussed here is a finite group. There are several previous articles that examine graphs formed from groups including Cayley graphs, G-graphs, coprime graphs, and identity graphs of dihedral groups.

2 Methodology: Notations and Definitions

The method used is literature study with the initial step of forming an identity graph of several cyclic groups then looking for general patterns of their properties, making conjectures and proving them. Before going into those steps, in this section we will discuss some basic concepts and definitions in group theory and graph theory.

Group Theory

These are some definitions in group theory [4] which will be used in the next section:

1. Group is a set with one binary operation on the set which fulfills associative properties, has an identity element, and each member of the group has an inverse.
2. Order of a group is the number of its elements. A finite group is a group of finite order. Let e is the identity element of a finite group G. Order of an element a in G is defined as the smallest natural number n such that $a^n = e$.
3. Let G be a group. A non-empty subset $H \subseteq G$ is called a subgroup of G if and only if H is also a group with the same operation defined in G [4].

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4. If G is a group and $a \in G$, then the set $\langle a \rangle = \{a^n : n \in Z\}$ is a subgroup of G, and $\langle a \rangle$ is called the cyclic subgroup generated by a. Group G is called a cyclic group if and only if $a \in G$ exists, such that $G = \langle a \rangle$ [4].

Related to order of groups and order of elements, we have these two important theorems in group theory [4]:

1. (Cauchy's Theorem) Let G be a finite group and p be a prime number. If p divides the order of G, then G has an element of order p.
2. (Lagrange's Theorem) If H is a subgroup of a finite group G, then order of H divides order of G.

Graph Theory

Graph G is a pair of finite sets (V, E), written with the notation $G\ (V, E)$, in which case V is a non-empty set of vertices and E is a non-empty set of edges connecting a pair of vertices or connect a vertex with the vertex itself. A graph G can be represented by a diagram, each vertex of G is represented by a dot or small circle while an edge connecting two vertices is represented by a curve connecting the corresponding vertices in the diagram.

3 Results and Discussion

In this section, we write our research results in terms of theorems and their proofs. Some illustrations of graphs are also presented. First, we need to understand the concept of identity graph of a group.

Definition [2]

Let G be a group. The identity graph of group G is a graph with the elements of group G as its vertices which satisfies these properties:

a) Two elements x, y in group G are connected by an edge if $xy = e$, with e is the identity element for group G.

b) Each element of G is connected by an edge with the identity element e.

To develop the previous research, we shall examine the identity graph of finite cyclic groups. There are two possibilities for the order of a finite cyclic group: it is an odd

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natural number, or it is an even natural number. The order may also be a prime number. We will examine the case of odd prime order first.

Theorem 1 [2]

If $G = \langle g \mid g^p = 1, p \neq 2 \rangle$ is a cyclic group of the pth order, where p is prime, then the identity graph formed by G consists of $(p-1)/2$ triangles.

Proof:

Let $G = \langle g \mid g\text{^}p = 1, p \neq 2 \rangle$ be a cyclic group of the pth order, where p is prime. Then G does not have a proper subgroup, according to Lagrange theorem. Therefore, there is no element in G having inverse which is itself; in other words, there is no $g^i \in G$ such that $\left(g^i\right)^2 = 1$. Suppose there is g^i in G such that $(g^i)\text{^}2 = 1$. Then, G has a subgroup $H = \{1, g^i\}$. This contradicts with the fact that G does not have a proper subgroup. As a result, G does not have element of the 2nd order. Using Cauchy theorem with $p \neq 2$ then G does not have element of the 2nd order.

For every(g^i) in G there is exactly one inverse of g^i that is g^j such that $g^i\, g^j = 1$ with i and j are positive integers and $i \neq j$. Because $g^i\, g^j = 1$ then $g^i\, g^j = g^{i+j} = 1 = g^p$ such that $p = i + j$ which is equivalent to $j = p - i$. Consequently, we can form the identity graph of G as illustrated by Figure 1.

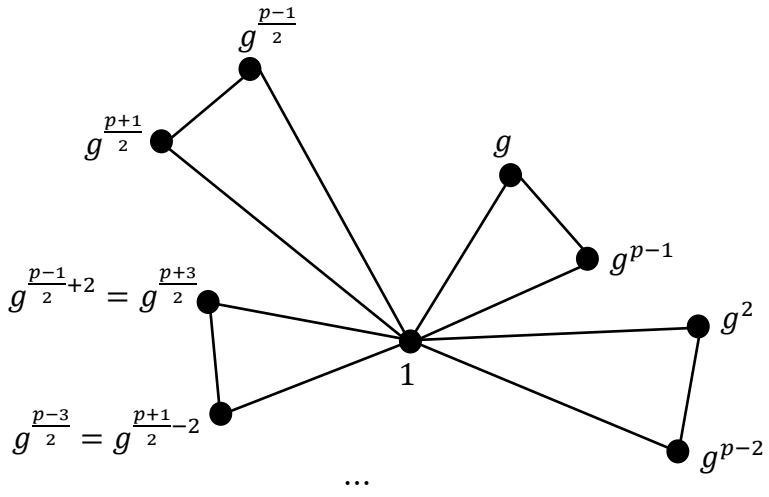


Figure 1. Identity graph of $G = \langle g \mid g^p = 1, p \neq 2 \rangle$

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We have seen a unique pattern of the identity graph of a finite cyclic group with odd prime order. We skip the case of even prime order since the only even prime number is 2, and the identity graph of group of order 2 looks too trivial and not so interesting. Now we look at more general case of finite cyclic groups of odd order.

Theorem 2 [2]

If G is a cyclic group with an odd order, then G has the identity graph G_i which can be formed by triangles without a unique edge.

Proof:

Let $G = \langle g \,|g^n = 1\rangle$ with n is an odd integer is a cyclic group with multiplication operation and is of the nth order. The elements of G are $\{1, g, g^2, g^3, g^4, \cdots, g^{n-1}\}$. We shall use the Cayley table given by Table 1 to show results of the operation for each element of G.

Table 1. Cayley table for Theorem 2.

	1	g	g^2	g^3	$\cdots$	g^{n-3}	g^{n-2}	g^{n-1}
1	1	g	g^2	g^3	$\cdots$	g^{n-3}	g^{n-2}	g^{n-1}
g	g	g^2	g^3	g^4	$\cdots$	g^{n-2}	g^{n-1}	1
g^2	g^2	g^3	g^4	g^5	$\cdots$	g^{n-1}	1	g
g^3	g^3	g^4	g^5	g^6	$\cdots$	1	g	g^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
g^{n-3}	g^{n-3}	g^{n-2}	g^{n-1}	1	$\cdots$	g^{n-6}	g^{n-5}	g^{n-4}
g^{n-2}	g^{n-2}	g^{n-1}	1	g	$\cdots$	g^{n-5}	g^{n-4}	g^{n-3}
g^{n-1}	g^{n-1}	1	g	g^2	$\cdots$	g^{n-4}	g^{n-3}	g^{n-2}

From the Cayley table above (Table 1) we can see that $gg^{n-1} = 1, g^2\, g^{n-2} = 1, \ldots, g^{n-1}g = 1$. Thus, for any non-identity element g, we have $g^k\, g^{n-k} = 1, for\ k \in$

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Z^+ and $k < n$. From the definition of identity graph part (iii), for any elements $a, b \in G, a \neq b, a \neq e, b \neq e$, there exists an edge which connects a to b if and only if $ab = ba = e$. Therefore, there exists an edge which connects $g^k to\ g^{n-k}$, and since for every $a \in G, a \neq e$ there exists an edge connecting a to e, then from the definition of identity graph part (ii), there exists an edge from g^k to 1. These will form a triangle connecting 1, g^k and g^{n-k}. Moreover, since n is an odd number, then $n = 2x + 1$ for any $x \in Z^+$, and the number of non-identity elements in G is even. So, those non-identity elements can be partitioned into two sets with same cardinality, where the inverse of an element in one set is in the other set and vice versa. Therefore, there is no single edge in identity graph of G.

Then the identity graph which corresponds with G is shown in Figure 2.

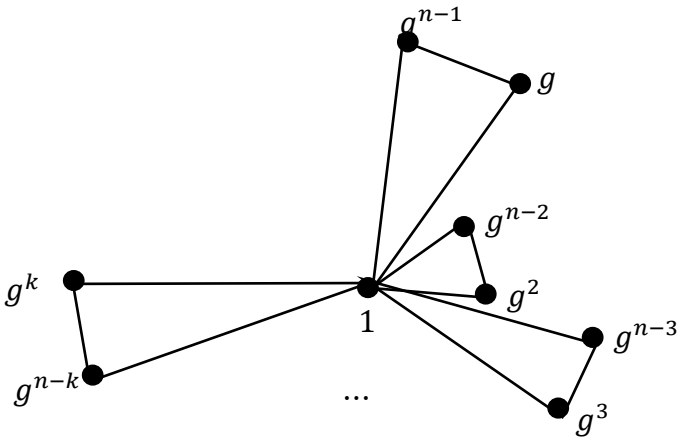


Figure 2. Identity graph of group $G = \{g \mid g^n = 1\}$, n is an odd number.

Based on previous theorems, we can conclude a characterization of the identity graph of cyclic group of odd order in the following Theorem 3.

Theorem 3 [2]

If $G = \langle g | g^n = 1 \rangle$ is a cyclic group of order n, where n is an odd number, then the identity graph G_i of G is formed by $(n-1)/2$ triangles.

Proof:

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This can be proved using Theorem 1 and Figure 1. Since the number of non-identity elements is even and those elements forms (n-1)/2 couples, where each couple is inverse to each other, then there will be (n-1)/2 triangles.

Similar with the result in the odd order case, we can obtain a characterization of the identity graph of cyclic group of even order case in the following Theorem 4. The proof is also using similar principle with the odd order case.

Theorem 4 [2]

If $G = \langle g | g^m = 1 \rangle$ is a cyclic group of order m where m is an even number, then its identity graph G_i has $(m-2)/2$ triangle and a single edge.

Proof:

Table 2. Cayley table for Theorem 4.

	1	g	g^2	g^3	$\cdots$	$g^{\frac{m}{2}}$	$\cdots$	g^{m-3}	g^{m-2}	g^{m-1}
1	1	g	g^2	g^3	$\cdots$	$g^{\frac{m}{2}}$	$\cdots$	g^{m-3}	g^{m-2}	g^{m-1}
g	g	g^2	g^3	g^4	$\cdots$	$g^{\frac{m}{2}+1}$	$\cdots$	g^{m-2}	g^{m-1}	1
g^2	g^2	g^3	g^4	g^5	$\cdots$	$g^{\frac{m}{2}+2}$	$\cdots$	g^{m-1}	1	g
g^3	g^3	g^4	g^5	g^6	$\cdots$	$g^{\frac{m}{2}+3}$	$\cdots$	1	g	g^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$g^{\frac{m}{2}}$	$g^{\frac{m}{2}}$	$g^{\frac{m}{2}+1}$	$g^{\frac{m}{2}+2}$	$g^{\frac{m}{2}+3}$	$\cdots$	1	$\cdots$	$g^{\frac{3m}{2}-3}$	$g^{\frac{3m}{2}-2}$	$g^{\frac{3m}{2}-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
g^{m-3}	g^{m-3}	g^{m-2}	g^{m-1}	1	$\cdots$	$g^{\frac{3m}{2}-3}$	$\cdots$	g^{m-6}	g^{m-5}	g^{m-4}
g^{m-2}	g^{m-2}	g^{m-1}	1	g	$\cdots$	$g^{\frac{3m}{2}-2}$	$\cdots$	g^{m-5}	g^{m-4}	g^{m-3}
g^{m-1}	g^{m-1}	1	g	g^2	$\cdots$	$g^{\frac{3m}{2}-1}$	$\cdots$	g^{m-4}	g^{m-3}	g^{m-2}

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Let $G = \langle g \,|g^m = 1\rangle$, m is an even number, be a cyclic multiplicative group and its order is m. Elements of G are $\{1, g, g^2, g^3, g^4, \cdots, g^{m-1}\}$. We will use the Cayley table given by Table 2 to show the operations between elements in G.

From the Cayley table above (Table 2) we can see that $g\,g^{m-1} = 1, g^2\,g^{m-2} = 1, \ldots, g^{m-1}\,g = 1$. However, for $g^{\frac{m}{2}}$ we have something different, that is $g^{\frac{m}{2}}\,g^{\frac{m}{2}} = 1$. Thus, for any non-identity elements other than $g^{\frac{m}{2}}$ we have $g^k\,g^{m-k} = 1$, for $k \in Z^+$ and $k < m$. From the definition of identity graph part (iii), for any elements $a, b \in G, a \neq b, a \neq e, b \neq e$, vertices a and b are adjacent if and only if $ab = ba = e$. This means the vertex $g\text{^}k$ is adjacent with vertex g^{m-k} and since for each $a \in G, a \neq e$ vertex a is adjacent with e based on definition of identity graph part (ii), then there exists an edge from g^k to 1. These form a triangle connecting 1, g^k dan $g\text{^}(m-k)$. In other side, for $g^{\frac{m}{2}}$ there will be only one edge connecting it, that is the edge which connects $g^{\frac{m}{2}}$ with 1. Moreover, since m is even, then $m = 2x$ for $x \in Z^+$ and the number of elements which is not the identity and not $g^{\frac{m}{2}}$ in G is even. Therefore, there will be $(m-2)/2$ triangles and a single edge. The identity graph G_i which corresponds with G is given in Figure 3.

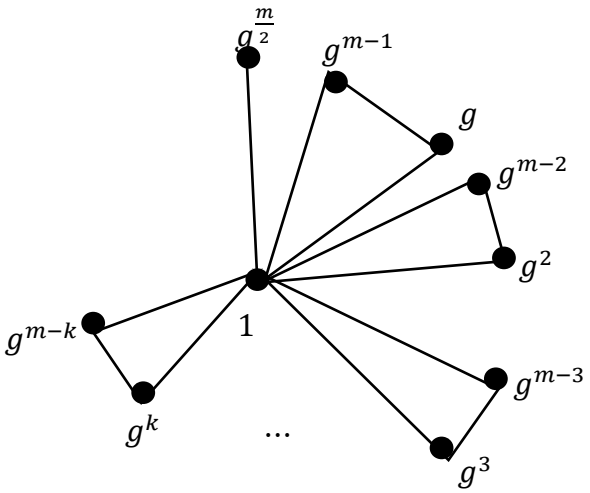


Figure 3. Identity graph of group $G = \langle g|\, g^m = 1\rangle$, n is an even number.

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We see that in the even order case, there exists exactly one single edge that does not form a triangle. This happens because in a finite cyclic group of even order, there exists exactly one element of order 2.

4 Conclusion

From what we have discussed, there are some conclusions about identity graph of finite cyclic groups as following:

a. Identity graph of a group is a way to represent the relations between elements of a group in a graph. In identity graph of a group, the identity element of a group is connected with any other elements and each non-identity element is connected with its inverse.

b. For cyclic group of odd order n, its identity graph consists of $(n-1)/2$ triangles. There is no single edge in this case since in such group there is no element of order 2.

c. For cyclic group of even order m, its identity graph consists of $(m-2)/2$ triangles and a single edge. The single edge connects the identity element to the only element of such group that has order 2.

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